Introduction to Matrix Analysis

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Numerical Analysis of Engineering Systems

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Outline

- Why matrices, basic definitions/terms
- Matrix multiplication
- Determinants
- Inverse of a Matrix
- Simultaneous linear equations: matrix from and Gaussian elimination solution
  - Matrix rank and its role in determining type of solutions (unique, infinite, none)
- Eigenvalues and eigenvectors

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Why Matrices

- Simplified notation for general statements about mathematical or engineering systems
- Have multiple components that are interrelated
  - Elements in a machine structure
  - Springs, masses and dampers in a vibrating system
- Matrix notation provides general relationships among components

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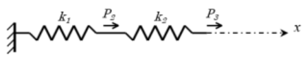
Matrix Example



- Simple linkage of two springs with spring constant k and displacements u_k at individual points
- Individual loads, P_k, related to individual displacements, u_k, by matrix equation shown below.

$$\boldsymbol{P}=\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}=\begin{bmatrix}k_1 & -k_1 & 0\\ -k_1 & k_1+k_2 & -k_2\\ 0 & -k_2 & k_2\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\boldsymbol{KU}$$

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Matrix Basics

$$\mathbf{A}=\begin{bmatrix}a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m}\\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2m}\\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3m}\\ \vdots & \vdots & \vdots & \ddots & & \vdots\\ \vdots & \vdots & \vdots & & \ddots & \vdots\\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nm}\end{bmatrix}$$

- Array of numbers with n rows and m columns
- Components are $a_{(row)(column)}$
- Size of matrix (n x m) or (n by m) is number of rows and columns

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Row and Column Matrices

- Matrices with only one row or only one column are called row or column matrices (sometimes called row or column vectors)
  - The row number 1 is usually dropped in row matrices as is the column number 1 in column matrices

$$\mathbf{r}=\begin{bmatrix}r_{11} & r_{12} & r_{13} & \cdots & \cdots & r_{1m}\end{bmatrix}=\begin{bmatrix}r_1 & r_2 & r_3 & \cdots & \cdots & r_m\end{bmatrix}$$

$$\mathbf{c}=\begin{bmatrix}c_{11}\\c_{21}\\c_{31}\\ \vdots\\ \vdots\\ c_{n1}\end{bmatrix}=\begin{bmatrix}c_1\\c_2\\c_3\\ \vdots\\ \vdots\\ c_n\end{bmatrix}$$

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More Matrix Basics

- Two matrices are equal (*e.g.*, **A** = **B**)
  - If both **A** and **B** have the same size (rows and columns)
  - If each component of **A** is the same as the corresponding component of **B** ($a_{ij} = b_{ij}$ for all i and j)
- A square matrix has the same number of rows and columns
- A diagonal matrix, **D**, has all zeros except for the principal diagonal

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Diagonal Matrix

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a_n \end{bmatrix}$$

- The diagonal matrix **A** is a square matrix with nonzero components only on the principal diagonal
- Components of **A** are $a_i\delta_{ij}$, where δ_{ij} is the Kroenecker delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

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Matrix Operations

- Can add or subtract matrices if they are the same size
  - **C** = **A** ± **B** only valid if **A**, **B**, and **C** have the same size (rows and columns)
  - Components of **C**, $c_{ij} = a_{ij} \pm b_{ij}$
- Multiplication by a scalar: **C** = x**A**
  - **C** and **A** have the same size (rows and columns)
  - Components of **C**, $c_{ij} = xa_{ij}$
  - For scalar division, **C** = **A**/x, $c_{ij} = a_{ij}/x$

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Null (**0**) and Unit (**I**) Matrices

- For any matrix, **A**, **A** + **0** = **0** + **A** = **A; IA** = **AI** = **A** and **0 A** = **A 0** = **0**
- The unit (or identity) matrix is a square matrix; the null matrix, which need not be square, is sometimes written $\mathbf{0}_{(nxm)}$

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & & & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix}$$

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Transpose of a Matrix

- Transpose of **A** denoted as $\mathbf{A}^T$
- Reverse rows and columns; for $\mathbf{B} = \mathbf{A}^T$
  - $b_{ij} = a_{ji}$
  - If **A** is (n x m), $\mathbf{B} = \mathbf{A}^T$ is (m by n)
    - In MATLAB the apostrophe is used to construct the transpose: `B = A'`

$$\mathbf{A} = \begin{bmatrix} 3 & 12 & -6 \\ 14 & -2 & 0 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 3 & 14 \\ 12 & -2 \\ -6 & 0 \end{bmatrix}$$

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Matrix Multiplication Preview

- Not an intuitive operation. Look at two coordinate transformations as example

$$y_1 = a_{11}x_1 + a_{12}x_2 \qquad z_1 = b_{11}y_1 + b_{12}y_2$$
$$y_2 = a_{21}x_1 + a_{22}x_2 \qquad z_2 = b_{21}y_1 + b_{22}y_2$$

- Substitute equations for y in terms of x into equations for z_1 and z_2

$$z_1 = b_{11}[a_{11}x_1 + a_{12}x_2] + b_{12}[a_{21}x_1 + a_{22}x_2]$$
$$z_2 = b_{21}[a_{11}x_1 + a_{12}x_2] + b_{22}[a_{21}x_1 + a_{22}x_2]$$

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Matrix Multiplication Preview II

- Rearrange last set of equations to get direct transformation from x to z

$$z_1 = [b_{11}a_{11} + b_{12}a_{21}]x_1 + [b_{11}a_{12} + b_{12}a_{22}]x_2 = c_{11}x_1 + c_{12}x_2$$
$$z_2 = [b_{21}a_{11} + b_{22}a_{21}]x_1 + [b_{21}a_{12} + b_{22}a_{22}]x_2 = c_{21}x_1 + c_{22}x_2$$

$$c_{11} = [b_{11}a_{11} + b_{12}a_{21}] \qquad c_{12} = [b_{11}a_{12} + b_{12}a_{22}]$$
$$c_{21} = [b_{21}a_{11} + b_{22}a_{21}] \qquad c_{22} = [b_{21}a_{12} + b_{22}a_{22}]$$

$$c_{ij} = \sum_{k=1}^{2} b_{ik}a_{kj} \qquad (i = 1,2;\ j = 1,2)$$

Matrix Multiplication Preview III

- Coefficients as matrix components

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}$$

C = **AB** if $c_{ij} = \sum_{k=1}^{2} b_{ik}a_{kj} \qquad (i = 1,2;\ j = 1,2)$

- Generalize this to any matrix size

Multiplying Matrices

- For general matrix multiplication, **C** = **AB**
  - **A** has n rows and p columns
  - **B** has p rows and m columns
  - **C** has n rows and m columns
  - **A** is left; **B** is right; **C** is product

$$c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj} \qquad (i = 1, n;\ j = 1, m)$$

- For **C** = **AB**, we get c_{ij} by adding products of terms in row i of **A** (left matrix) by terms in column j of **B** (right matrix)
- $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + a_{i4}b_{4j} + \ldots$
- In general, **AB** ≠ **BA**

General Matrix Multiplication

- For matrix multiplication, **C** = **AB**
  - **A** has n rows and p columns
  - **B** has p rows and m columns
  - **C** has n rows and m columns
  - **A** is left; **B** is right; **C** is product

$$c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj} \qquad (i = 1, n;\ j = 1, m)$$

- Example

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -6 \\ 4 & -2 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \\ 6 & 1 \end{bmatrix}$$

$(\mathbf{AB})_{ij}$ is product of row i of **A** times column j of **B**

$$\mathbf{AB} = \begin{bmatrix} 3(3) + 0(1) - 6(6) & 3(4) + 0(2) - 6(1) \\ 4(3) - 2(1) + 0(6) & 4(4) - 2(2) + 0(1) \end{bmatrix} = \begin{bmatrix} -27 & 6 \\ 10 & 12 \end{bmatrix}$$

Matrix Multiplication Exercise

- Consider the following matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & -3 & 1 \end{bmatrix}$$

- Can you find **AB**, **BA** or both?
- We can find **AB**, because **A** has three columns and **B** has three rows
- We cannot find **BA** because **B** has four columns and **A** has three rows
- Next chart starts the process of finding all components of **AB**

Matrix Multiplication Exercise II

- What is the size of **C** = **AB**?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & -3 & 1 \end{bmatrix}$$

- **C** = **AB** has three rows (like **A**) and four columns (like **B**)
- What is c_{11}?
- $c_{11} = (1)(0) + (2)(1) + (3)(2) = 8$

Matrix Multiplication Exercise III

- Find c_{11}, c_{12}, c_{13}, and c_{14} in **C** = **AB**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & -3 & 1 \end{bmatrix}$$

- c_{11} = (1)(0) + (2)(1) + (3)(2) = 8
- c_{12} = (1)(2) + (2)(3) + (3)(1) = 11
- c_{13} = (1)(-2 … = -9
- c_{14} = (1)(0) … 3

$$\mathbf{C} = \begin{bmatrix} 8 & 11 & -9 & 3 \\ & & & \\ & & & \end{bmatrix}$$

Matrix Multiplication Exercise IV

- Find c_{21}, c_{22}, c_{23}, and c_{24} in **C** = **AB**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & -3 & 1 \end{bmatrix}$$

- c_{21} = (0)(0) + (-1)(1) + (4)(2) = 7
- c_{22} = (0)(2) + (-1)(3) + (4)(1) = 1
- c_{23} = (0)(- … 3) = -13
- c_{24} = (0)(… = 4

$$\mathbf{C} = \begin{bmatrix} 8 & 11 & -9 & 3 \\ 7 & 1 & -13 & 4 \\ & & & \end{bmatrix}$$

Matrix Multiplication Exercise V

- Find c_{31}, c_{32}, c_{33}, and c_{34} in **C** = **AB**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & -3 & 1 \end{bmatrix}$$

- c_{31} = (1)(0) + (1)(1) + (0)(2) = 1
- c_{32} = (1)(2) + (1)(3) + (0)(1) = 5
- c_{33} = (1)(- …) = -1
- c_{34} = (1)(… = 0

$$\mathbf{C} = \begin{bmatrix} 8 & 11 & -9 & 3 \\ 7 & 1 & -13 & 4 \\ 1 & 5 & -1 & 0 \end{bmatrix}$$

Matrix Multiplication Results

- Solution for matrix product **C** = **AB**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 1 & -3 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 8 & 11 & -9 & 3 \\ 7 & 1 & -13 & 4 \\ 1 & 5 & -1 & 0 \end{bmatrix}$$

MATLAB/Excel Matrices

- Have seen term-by-term operations on arrays, A+B, A-B, A.*B, A./B and A.^n
- Matrix operations are given by following: A+B, A-B, A*B, A/B, A\B and A^n
- MATLAB $\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $\boldsymbol{A}.\,^{\wedge}\mathbf{2} = \begin{bmatrix} 1 & 9 \\ 4 & 16 \end{bmatrix}$
- A^2: $\boldsymbol{A}^{\mathbf{2}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix}$
- Excel has mmult array function to multiply two arrays

Determinants

- Looks like a matrix but isn't a matrix
- A square array of numbers with a rule for computing a single value for the array

Example at right shows calculation of Det(**A**), the determinant of 3 x 3 matrix **A**

$$Det \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}$$

More Determinants

- Useful in obtaining algebraic expressions for matrix operations, but not useful for numerical computation
  - Equation for 2 by 2 determinant

$$Det\,\mathbf{A} = Det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

- An n x n determinant has n^2 Minors, M_{ij}, obtained by deleting row i and column j
- Cofactors, $A_{ij} = (-1)^{i+j}M_{ij}$ used in general expressions for determinants

General Rule for Determinants

- Any size determinant can be evaluated by any of the following equations

$$Det\,\mathbf{A} = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}M_{ij} = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}M_{ij} = \sum_{i=1}^{n}a_{ij}A_{ij} = \sum_{j=1}^{n}a_{ij}A_{ij}$$

- Can pick **any** row **or any** column
- Choose row or column with most zeros to simplify calculations
- Can apply equation recursively; evaluate a 5 x 5 determinant as a sum of 4 x 4 determinants then get 4 x 4's in terms of 3 x 3's

Determinant Behavior

- A determinant is zero if any row or any column contains all zeros.
- If one row or one column of a determinant is multiplied by a constant, k, the value of the determinant is multiplied by the same constant.
  - Note the implication for matrices: if a matrix is multiplied by a constant, k, then each matrix element is multiplied by k. If **A** is an n x n matrix, Det(k**A**) = k^nDet(**A**).

Determinant Behavior II

- If one row (or one column) of a determinant is replaced by a linear combination of that row (or column) with another row (or column), the value of the determinant is not changed.
- If two rows (or two columns) of a determinant are linearly dependent the value of the determinant is zero.

Determinant Behavior III

- The determinant of the product of two matrices, **A** and **B** is the product of the determinants of the individual matrices: Det(**AB**) = Det(**A**) Det(**B**).
- The determinant of transposed matrix is the same as the determinant of the original matrix: Det($\mathbf{A}^T$) = Det(**A**).
- MATLAB: det(A) gives Det(**A**)
- Excel: mdeterm; video for all Excel matrix
  - http://www.youtube.com/watch?v=gY81Jg7jyc4

Inverse of a Matrix

- For a ***square*** matrix, **A**, an inverse matrix, $\mathbf{A}^{-1}$ ***may exist*** such that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- For the algebraic equation ax = b, x = a^{-1}b
- For the matrix equation **Ax** = **b**, **x** = $\mathbf{A}^{-1}\mathbf{b}$
- Just as x = a^{-1}b is not valid if a = 0, **x** = $\mathbf{A}^{-1}\mathbf{b}$ is not valid if $\mathbf{A}^{-1}$ does not exist
  - A^{-1} does not exist if Det**A** = 0
- The inverse is an important concept in analysis of linear systems, but is not used in computational work

Formula for Inverse of a Matrix

- Find the components of **B** = $\mathbf{A}^{-1}$, b_{ij}, from determinant of **A** and its cofactors

$$If\ \mathbf{B} = \mathbf{A}^{-1}, \qquad b_{ij} = \frac{A_{ji}}{Det(\mathbf{A})} = (-1)^{i+j}\frac{M_{ji}}{Det(\mathbf{A})}$$

- Use to get algebraic equations for components of inverse matrix
- Matrix computations, if necessary, obtain components by alternative numerical algorithms

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Inverse of 2 x 2 Matrix

$$If\ \mathbf{B} = \mathbf{A}^{-1}, \qquad b_{ij} = \frac{A_{ji}}{Det(\mathbf{A})} = (-1)^{i+j}\frac{M_{ji}}{Det(\mathbf{A})}$$

$$b_{11} = (-1)^{1+1}\frac{M_{11}}{Det(\mathbf{A})} = \frac{a_{22}}{Det(\mathbf{A})} \qquad b_{12} = (-1)^{1+2}\frac{M_{21}}{Det(\mathbf{A})} = -\frac{a_{12}}{Det(\mathbf{A})}$$

$$b_{21} = (-1)^{2+1}\frac{M_{12}}{Det(\mathbf{A})} = -\frac{a_{21}}{Det(\mathbf{A})} \qquad b_{22} = (-1)^{2+2}\frac{M_{22}}{Det(\mathbf{A})} = \frac{a_{11}}{Det(\mathbf{A})}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

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Inverse of 3 x 3 Matrix

$$\boxed{Apply \quad \mathbf{B} = \mathbf{A}^{-1}, \quad b_{ij} = (-1)^{i+j}\frac{M_{ji}}{Det(\mathbf{A})}}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} =$$

$$\frac{\begin{bmatrix} (a_{22}a_{33} - a_{32}a_{23}) & (a_{32}a_{13} - a_{33}a_{12}) & (a_{12}a_{23} - a_{22}a_{13}) \\ (a_{31}a_{23} - a_{33}a_{21}) & (a_{11}a_{33} - a_{31}a_{13}) & (a_{21}a_{13} - a_{11}a_{23}) \\ (a_{21}a_{32} - a_{31}a_{22}) & (a_{31}a_{12} - a_{11}a_{32}) & (a_{11}a_{22} - a_{21}a_{12}) \end{bmatrix}}{\begin{pmatrix} a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ -a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13} \end{pmatrix}}$$

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MATLAB/Excel Inverse

- MATLAB: use B = inv(A) to get **B** = $\mathbf{A}^{-1}$
- MATLAB can use C=B/A to get **C** = $\mathbf{A}^{-1}\mathbf{B}$
- Excel function minverse computes inverse
  - Select empty cell area same size as matrix
  - Enter formula =minverse(<matrix cells>)
  - Press control+shift+enter

	A	B	C	D	E	F
1	1	2	3	-2	1	0
2	1	3	-1	1	0	1

- Original matrix in cells A1:B2
- Array formula =minverse(A1:B2) in cells C1:D2
- Array formula =mmult(A1:B2,C1:D2) in cells E1:F2

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Simultaneous Equation Basics

- A set of simultaneous linear algebraic equations may have
  - A single (unique) solution
  - No solution
  - An infinite number of solutions
- A linear combination of any two equations can replace one of the equations and not change the solution

Eqn I $3x_1 + 5x_2 = 13$ $\qquad 6x_1 + 11x_2 = 28$ *(Eqn II) –*

Eqn II $6x_1 + 11x_2 = 28$ $\qquad -2[3x_1 + 5x_2 = 13]$ *2 * (Eqn I)*

$$x_2 = 2$$

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Simultaneous Equations

- The second column is an equivalent set of equations that is a linear combination of the equations in the first column

$3x_1 + 5x_2 = 13$ $6x_1 + 11x_2 = 28$	$3x_1 + 5x_2 = 13$ $x_2 = 2$	$x_1 = 1$ $x_2 = 2$
$3x_1 + 5x_2 = 13$ $6x_1 + 10x_2 = 26$	$3x_1 + 5x_2 = 13$ $0 = 0$	$x_1 = a \quad (any\ a)$ $x_2 = \frac{13 - 3a}{5}$
$3x_1 + 5x_2 = 13$ $6x_1 + 10x_2 = 25$	$3x_1 + 5x_2 = 13$ $0 = -1$	*No solution*

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Getting to a Matrix Form

- Example of 3 equations (3 unknowns)
  - 3x + 7y – 3z = 8
  - 2x – 4y + z = -3
  - 8x + 6y – 2z = 14
- How can we develop a general notation for N equations in N unknowns?
  - Call variables x_1, x_2, x_3 etc.
  - Call right hand side b_1, b_2, b_3, etc.
  - Call top row coefficients a_{11}, a_{12}, a_{13}, etc.
    - Coefficient of x_j in equation i is a_{ij}

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Standard Form

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1N-1}x_{N-1} + a_{1N}x_N = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2N-1}x_{N-1} + a_{2N}x_N = b_2$
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3N-1}x_{N-1} + a_{3N}x_N = b_3$
… … … … … … … … … … … … … … … … … …
$a_{N-1,1}x_1 + a_{N-1,2}x_2 + \ldots + a_{N-1,N}x_N = b_{N-1}$
$a_{N1}x_1 + a_{N2}x_2 + a_{N3}x_3 + \ldots + a_{NN}x_N = b_N$

- Usual subscripts on a are $a_{row,column}$
- Row is equation and x column is unknown
- N can be any number

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Compact Standard Forms

- **Ax** = **b** $\qquad \sum_{j=1}^{N} a_{ij}x_j = b_i \qquad i = 1, \ldots, N$
- Equations defined by data: N, a_{ij}, and b_i
- Summation is same as matrix multiplication formula
- a_{ij} coefficients are a matrix, **A**
- Right-hand side, **b**, and unknowns, **x**, are column vectors

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Example in Standard Form

- Previous example of 3 equations (N = 3)
  - 3x + 7y – 3z = 8
  - 2x – 4y + z = -3
  - 8x + 6y – 2z = 14
- In standard form:
  - x is x_1, y is x_2, and z is x_3
  - $a_{11} = 3$, $a_{12} = 7$, $a_{13} = -3$, b_1, = 8
  - $a_{21} = 2$, $a_{22} = -4$, $a_{23} = 1$, b_2, = -3
  - $a_{31} = 8$, $a_{32} = 6$, $a_{33} = -2$, b_3, = 14

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Example in Standard Form

- Previous example of N = 3 equations
  - 3x + 7y – 3z = 8
  - 2x – 4y + z = -3
  - 8x + 6y – 2z = 14
- As **Ax** = **b**

$$\begin{bmatrix} 3 & 7 & -3 \\ 2 & -4 & 1 \\ 8 & 6 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ 14 \end{bmatrix}$$

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Example in Standard Form

- Multiply **A** and **x** as shown below

$$\begin{bmatrix} 3 & 7 & -3 \\ 2 & -4 & 1 \\ 8 & 6 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 + 7x_2 - 3x_3 \\ 2x_1 - 4x_2 + 1x_3 \\ 8x_1 + 6x_2 - 2x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ 4 \end{bmatrix}$$

- Final equal sign for two (3x1) matrices gives original form of three simultaneous equations
  - 3x + 7y – 3z = 8
  - 2x – 4y + z = -3
  - 8x + 6y – 2z = 14

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Solve **Ax** = **b** in MATLAB/Excel

- MATLAB could use **x** = inv(**A**) * **b**
  - Preferred approach is **x** = **b****A** which is faster and more accurate for solving **Ax**=**b**
- Excel: Select column for **x** solution then enter array formula:
  - mmult(minverse(<**A** cells>), <**b** cells>)

	A	B	C	D	E	F
1		A		b	x	check
2	3	7	-3	8	1	-7.10543E-15
3	2	-4	1	-3	2	8.88178E-16
4	8	6	-2	14	3	-5.32907E-15

Array formula in f2:f4 is: =mmult(a2:c4, e2:e4)-d2:d4

Array formula in e2:e4 is: =mmult(minverse(a2:c4),d2:d4)

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Solving **Ax** = **b**

- Know A (all the a_{ij}) and b (all b_i)
- Want x (all the unknowns x_j)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

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General System for **Ax** = **b**

(n x m) (m x 1) (n x 1)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

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n equations and m unknowns?

- How can this be? We expect m = n
- First we have to see if the n equations are really independent equations
- Systems for m > n have an infinite number of solutions
- Systems for n > m can be solved in a least squares sense
  - Provide solution that has least error in solution: minimize $\sum_{i=1}^{n}\left(b_i - \sum_{j=1}^{m} A_{ij}x_j\right)^2$

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Gauss Elimination

- Practical tool for obtaining solutions
- Analytical tool for determining linear dependence or independence
- Basic idea is to manipulate the equations (or data) to make them easier to solve without changing the results
- Systematically create zeros in lower left part of the equations (or data)

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Upper Triangular Form (UTF)

- Convert original set of equations to UTF

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \cdots & \alpha_{1n-1} & \alpha_{1n} \\ 0 & \alpha_{22} & \alpha_{23} & \cdots & \cdots & \alpha_{2n-1} & \alpha_{2n} \\ 0 & 0 & \alpha_{33} & \cdots & \cdots & \alpha_{3n-1} & \alpha_{3n} \\ \vdots & \vdots & \vdots & \ddots & & & \\ \vdots & \vdots & \vdots & & \ddots & & \\ 0 & 0 & 0 & \cdots & \cdots & \alpha_{n-1n-1} & \alpha_{n-1n} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \alpha_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

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Gauss Elimination III

- Upper triangular form on previous slide is easily solved by back substitution
- $x_n = \beta_n/\alpha_{nn}$
- $x_{n-1} = (\beta_{n-1} - \alpha_{n-1\,n}x_n)/\alpha_{n-1\,n-1}$, *et cetera*
- General equation for back substitution

$$x_i = \frac{\beta_i - \sum_{j=i+1}^{n} \alpha_{ij} x_j}{\alpha_{ii}} \qquad i = n, n-1, n-2, \ldots, 1$$

Convention: If lower index is greater than upper index in Σ operator, do not execute sum

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Gauss Elimination Algorithm

- How do we get the upper triangular form?
- Work on ***augmented matrix***

$$[\mathbf{A}, \mathbf{b}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2m} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3m} & b_3 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nm} & b_n \end{bmatrix}$$

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Gauss Elimination Algorithm II

- Repeatedly replace rows by linear combination of two rows that produces a zero in a desired row/column combination
- First step: make all of column 1 below row 1 zero (sets all $a_{r1} = 0$ except a_{11})
- Second step: make all of column 2 below row 2 zero (sets all $a_{r2} = 0$ for r > 2)
- Continue this kind of step for all rows except the last row
  - Row being subtracted is called "pivot row"

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General Gauss Elimination

- Use each row from row 1 to row n-1 as the "pivot" row
  - Diagonal element on pivot row is $a_{pivot,pivot}$
  - For each row (row r) below the pivot row
    - Multiply that row by $a_{row,pivot}/a_{pivot,pivot}$
    - Subtract result from row r to make $a_{row,pivot} = 0$
    - Operation requires subtraction for each column of **A** right of pivot column and for **b**
  - Repeat for each row below pivot row
- Repeat for rows 1 to n-1 as pivot rows
- Replace existing array with results of new operations

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General Gauss Elimination II

- Operations with row 2 as pivot row
- Replace RowR by {RowR – (a_{R2}/a_{22})Row2}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} & b_1 \\ 0 & a_{22} & a_{23} & \cdots & \cdots & a_{2m} & b_2 \\ 0 & a_{32} - \frac{a_{32}}{a_{22}} a_{22} & a_{33} - \frac{a_{32}}{a_{22}} a_{23} & \cdots & \cdots & a_{3m} - \frac{a_{32}}{a_{22}} a_{2m} & b_3 - \frac{a_{32}}{a_{22}} b_2 \\ 0 & a_{42} - \frac{a_{42}}{a_{22}} a_{22} & a_{43} - \frac{a_{42}}{a_{22}} a_{23} & \ddots & & a_{4m} - \frac{a_{42}}{a_{22}} a_{2m} & b_4 - \frac{a_{42}}{a_{22}} b_2 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & a_{n2} - \frac{a_{n2}}{a_{22}} a_{22} & a_{n3} - \frac{a_{n2}}{a_{22}} a_{23} & \cdots & \cdots & a_{nm} - \frac{a_{n2}}{a_{22}} a_{2m} & b_n - \frac{a_{n2}}{a_{22}} b_2 \end{bmatrix}$$

$$a_{row,column} \leftarrow a_{row,column} - \frac{a_{row,pivot}}{a_{pivot,pivot}} a_{pivot,column}$$

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Gauss Pseudocode

$$\text{For } pivot = 1 \text{ to } n-1 \left\{ \text{For } row = pivot+1 \text{ to } n \left\{ \begin{array}{l} \left[\text{For } column = pivot+1 \text{ to } n \left(a_{row,column} \leftarrow a_{row,column} - \frac{a_{row,pivot}}{a_{pivot,pivot}} a_{pivot,column} \right) \right. \\ \left. b_{row} \leftarrow b_{row} - \frac{a_{row,pivot}}{a_{pivot,pivot}} b_{pivot} \right] \end{array} \right\} \right.$$

Operations on b_{row} are the same as operations would be on $a_{row,n+1}$

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Solution Details

- Solve the set of equations on the right

$$
\begin{aligned}
2x_1 - 4x_2 - 26x_3 &= -34 \; (i) \\
-3x_1 + 2x_2 + \quad 9x_3 &= \quad 13 \; (ii) \\
7x_1 + 3x_2 + \quad 8x_3 &= \quad 14 \; (iii)
\end{aligned}
$$

- Subtract –3/2 times (i) from equation (ii) and 7/2 times (i) from (iii)

$$\boxed{\left[-3-\left(\frac{-3}{2}\right)2\right]x_1} + \left[2-\left(\frac{-3}{2}\right)(-4)\right]x_2 + \left[9-\left(\frac{-3}{2}\right)(-26)\right]x_3 = \left[13-\left(\frac{-3}{2}\right)(-34)\right]$$

$$\boxed{\left[7-\left(\frac{7}{2}\right)2\right]x_1} + \left[3-\left(\frac{7}{2}\right)(-4)\right]x_2 + \left[8-\left(\frac{7}{2}\right)(-26)\right]x_3 = \left[14-\left(\frac{7}{2}\right)(-34)\right]$$

Unnecessary operations

More Details

- Result from first set of operations

$$
\begin{aligned}
2x_1 - 4x_2 - 26x_3 &= -34 \\
0x_1 - 4x_2 - 30x_3 &= -38 \\
0x_1 + 17x_2 + 99x_3 &= 133
\end{aligned}
$$

- Subtract 17/(-4) times (ii) from (iii)

$$x_2\left[17-\left(\frac{17}{-4}\right)(-4)\right] + x_3\left[99-\left(\frac{17}{-4}\right)(-30)\right] = \left[133-\left(\frac{17}{-4}\right)(-38)\right]$$

- Final upper-triangular form

$$
\begin{aligned}
2x_1 - 4x_2 - 26x_3 &= -34 \\
0x_1 - 4x_2 - 30x_3 &= -38 \\
0x_1 + 0x_2 - \frac{57}{2}x_3 &= -\frac{57}{2}
\end{aligned}
$$

Back Substitution

- Final upper-triangular form

$$
\begin{aligned}
2x_1 - 4x_2 - 26x_3 &= -34 \\
0x_1 - 4x_2 - 30x_3 &= -38 \\
0x_1 + 0x_2 - \frac{57}{2}x_3 &= -\frac{57}{2}
\end{aligned}
$$

- We see that $x_3 = 1$ and back substitution gives x_2 and x_1 as follows

$$-4x_2 - 30x_3 = -38$$

$$x_2 = \frac{-38 + 30(x_3)}{-4} = \frac{-38 + 30(1)}{-4}$$

$$x_2 = 2$$

$$2x_1 - 4x_2 - 26x_3 = -34$$

$$x_1 = \frac{-34 + 4x_2 + 26x_3}{2}$$

$$x_1 = \frac{-34 + 4(2) + 26(1)}{2} = 0$$

Do we have a solution to **Ax** = **b**?

- Answer to question based on the rank which is defined as the number of linearly independent **rows or columns**
- Use Gauss elimination to determine rank
  - Use Gauss elimination to convert matrix to upper-triangular form
  - In this form, **rank is number of rows with non-zero coefficients**
  - This is sometimes called row-echelon form
- MATLAB has function rank(matrix)

What is a row-echelon form?

- Apply Gauss elimination to get
  - All zeros below row one in column one
  - All zeros below row two in column two
  - Keep this up until you get to the final row or until there are no more rows with nonzeros
- Count number of rows that are ***not*** all zeros; this is the rank
- This is way to determine linear independence of a set of vectors

What is the rank of each matrix?

$$\begin{bmatrix} 6 & 0 & 2 & 0 & 0 & 0 & 1 & 6 \\ 0 & 1 & 7 & 8 & 6 & 2 & 8 & 4 \\ 0 & 0 & 2 & 0 & 3 & 5 & 8 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 0 & 6 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 6 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 7 & 8 & 6 & 2 \\ 0 & 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 9 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Finding Rank

- What is rank of matrix, **A**?
- What is maximum possible value for rank? *2*
- Lower right matrix is result of applying Gauss elimination to **A**
- What is its rank?
- Rank **A** = 1

$$\mathbf{A} = \begin{bmatrix} 1 & -12 \\ -6 & 72 \\ 13 & -156 \\ -7 & 84 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -12 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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Solutions to **Ax** = **b**

- For a system of n unknowns
- If rank **A** = rank [**A b**] = n there is a unique solution
- If rank **A** = rank [**A b**] < n there are an infinite number of solutions
- If rank **A** ≠ rank [**A b**] there are no solutions

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Three Examples

Original Equations	Triangularized Set	Solutions
$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $7x_1 + 3x_2 + 8x_3 = -13$	$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $50.5x_3 = 50.5$	$x_1 = 0$ $x_2 = -7$ $x_3 = 1$
$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $-2x_1 + 10x_2 + 61x_3 = -9$	$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $0 = 0$	$x_1 = -8 - 8\alpha$ $x_2 = -2.5 - 4.5\alpha$ $x_3 = \alpha$
$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $-2x_1 + 10x_2 + 61x_3 = -8$	$x_1 - 4x_2 - 26x_3 = 2$ $2x_2 + 9x_3 = -5$ $0 = 1$	No solution

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First Example Rank

Original
$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ 7 & 3 & 8 \end{bmatrix} \quad [\mathbf{A\,b}] = \begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ 7 & 3 & 8 & -13 \end{bmatrix}$$

Row-echelon form
$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ 0 & 0 & 50.5 \end{bmatrix} \quad [\mathbf{A\,b}] = \begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ 0 & 0 & 50.5 & 50.5 \end{bmatrix}$$

Here we see that rank A = rank [A b] = number of unknowns = 3 so we have a unique solution

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Second Example Rank

Original
$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ -2 & 10 & 61 \end{bmatrix} \quad [\mathbf{A\,b}] = \begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ -2 & 10 & 61 & -9 \end{bmatrix}$$

Row-echelon form
$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A\,b}] = \begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank **A** = rank [**A b**] = 2 which is less than the number of unknowns (3) so we have an infinite number of solutions

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Third Example Rank

Original
$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ -2 & 10 & 61 \end{bmatrix} \quad [\mathbf{A\,b}] = \begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ -2 & 10 & 61 & -8 \end{bmatrix}$$

Row-echelon form
$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -26 \\ 0 & 2 & 9 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A\,b}] = \begin{bmatrix} 1 & -4 & -26 & 2 \\ 0 & 2 & 9 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, rank **A** = 2 ≠ rank [**A b**] = 3; therefore we have no solutions

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Homogenous Equations

- If **b** = **0**, *i.e.*, each $b_i = 0$, we automatically have rank **A** = rank[**A b**] so we have a solution
- If this rank equals the number of unknowns, we have a unique solution, **x** = **0**
- If this rank is less than the number of unknowns we have an infinite number of solutions

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Homogenous Equation Example

Equations $a_{11} = -1$ A matrix

$$\begin{aligned} -x_1 - 4x_2 + 3x_3 &= 0 \\ -4x_1 + 11x_2 - 6x_3 &= 0 \\ x_1 - 8x_2 + 5x_3 &= 0 \end{aligned} \qquad \mathbf{A} = \begin{bmatrix} -1 & -4 & 3 \\ -4 & 11 & -6 \\ 1 & -8 & 5 \end{bmatrix}$$

Original [A b] matrix

$$[\mathbf{A\,b}] = \begin{bmatrix} -1 & -4 & 3 & 0 \\ -4 & 11 & -6 & 0 \\ 1 & -8 & 5 & 0 \end{bmatrix}$$

Row-echelon form

$$[\mathbf{A\,b}] = \begin{bmatrix} -1 & -4 & 3 & 0 \\ 0 & 27 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank A = rank [A b] = 2 < unknowns = 3 so there are infinite solutions

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Homogenous Equation Example II

Equations $a_{11} = +1$ A matrix

$$\begin{aligned} x_1 - 4x_2 + 3x_3 &= 0 \\ -4x_1 + 11x_2 - 6x_3 &= 0 \\ x_1 - 8x_2 + 5x_3 &= 0 \end{aligned} \qquad \mathbf{A} = \begin{bmatrix} 1 & -4 & 3 \\ -4 & 11 & -6 \\ 1 & -8 & 5 \end{bmatrix}$$

Original [A b] matrix

$$[\mathbf{A\,b}] = \begin{bmatrix} 1 & -4 & 3 & 0 \\ -4 & 11 & -6 & 0 \\ 1 & -8 & 5 & 0 \end{bmatrix}$$

Row-echelon form

$$[\mathbf{A\,b}] = \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & -5 & 6 & 0 \\ 0 & 0 & 2.8 & 0 \end{bmatrix}$$

Rank A = rank [A b] = unknowns = 3 so there is a unique solution (**x** = **0**)

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Rank and Determinants

- Determinant rank, like matrix rank, is the number of linearly independent rows or columns.
- Two equivalent statements: a determinant is zero if
  - its rows are linearly dependent
  - the size of a determinant is larger than its rank

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Practical Determinant Evaluation

- Use Gauss elimination and find the product of the elements on the diagonal
  - A determinant does not change if one row is replaced by a linear combination of that row with another row
  - Gauss elimination converts a determinant into upper-triangular form without changing its value
  - The determinant of an upper-triangular array is the product of the components on the principal diagonal

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Upper Triangular Determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{55} \end{vmatrix}$$

$$= a_{11}(-1)^{2+2} a_{22} \begin{vmatrix} a_{33} & a_{34} & a_{35} \\ 0 & a_{44} & a_{45} \\ 0 & 0 & a_{55} \end{vmatrix} = a_{11} a_{22} (-1)^{3+3} a_{33} \begin{vmatrix} a_{44} & a_{45} \\ 0 & a_{55} \end{vmatrix}$$

$$= a_{11} a_{22} a_{33} a_{44} a_{55} = \prod_{k=1}^{5} a_{kk}$$

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Determinant Sign

- Gauss elimination uses row swapping to reduce round-off error
- If two rows in a determinant are swapped, the determinant sign changes

$$Det\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = (1)(4) - (3)(2) = -2$$

$$Det\begin{bmatrix}3 & 4\\ 1 & 2\end{bmatrix} = (3)(2) - (1)(4) = 2$$

- In Gauss elimination keep a count of the row swaps, nSwaps; find determinant from diagonalized array by the formula

$$(-1)^{nSwaps}\prod_{k=1}^{N} a_{kk}$$

Homogenous Infinite Solutions

Example $a_{11} = -1$ **A** matrix

$$\begin{aligned}-x_1 - 4x_2 + 3x_3 &= 0\\ -4x_1 + 11x_2 - 6x_3 &= 0\\ x_1 - 8x_2 + 5x_3 &= 0\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix}-1 & -4 & 3\\ -4 & 11 & -6\\ 1 & -8 & 5\end{bmatrix}$$

Row-echelon form

$$\tilde{\mathbf{A}} = \begin{bmatrix}-1 & -4 & 3\\ 0 & 27 & -18\\ 0 & 0 & 0\end{bmatrix}$$

$$\begin{aligned}Det\ \tilde{\mathbf{A}} = Det\ \mathbf{A} &= (-1)(11)(5) +\\ &(-4)(-8)(3) + (1)(-4)(-6)\\ &-(1)(11)(3) - (-4)(-4)(5) - (-1)(-8)(-6)\\ &= -55 + 96 + 24 - 33 - 80 + 48 = 0\end{aligned}$$

Det **A** = 0 => solution of **x** ≠ **0** may exist

Homogenous Infinite Solutions

Example $a_{11} = +1$ **A** matrix

$$\begin{aligned}x_1 - 4x_2 + 3x_3 &= 0\\ -4x_1 + 11x_2 - 6x_3 &= 0\\ x_1 - 8x_2 + 5x_3 &= 0\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix}1 & -4 & 3\\ -4 & 11 & -6\\ 1 & -8 & 5\end{bmatrix}$$

$$\begin{aligned}Det\ \mathbf{A} &= (1)(11)(5) +\\ &(-4)(-8)(3) + (1)(-4)(-6)\\ &-(1)(11)(3) - (-4)(-4)(5) - (1)(-8)(-6)\\ &= 55 + 96 + 24 - 33 - 80 - 48 = 14\end{aligned}$$

Row-echelon form

$$\tilde{\mathbf{A}} = \begin{bmatrix}1 & -4 & 3\\ 0 & -5 & 6\\ 0 & 0 & -2.8\end{bmatrix}$$

$$Det\ \tilde{\mathbf{A}} = (1)(-5)(-2.8) = 14$$

Det **A** ≠ 0 and **b** = **0** means **x** = **0**

Cramer's Rule

- You may like to use this for small systems of equations

$$x_1 = \frac{\begin{vmatrix}b_1 & a_{12} & a_{13}\\ b_2 & a_{22} & a_{23}\\ b_3 & a_{32} & a_{33}\end{vmatrix}}{\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}} \quad x_2 = \frac{\begin{vmatrix}a_{11} & b_1 & a_{13}\\ a_{21} & b_2 & a_{23}\\ a_{31} & b_3 & a_{33}\end{vmatrix}}{\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}} \quad x_3 = \frac{\begin{vmatrix}a_{11} & a_{12} & b_1\\ a_{21} & a_{22} & b_2\\ a_{31} & a_{32} & b_3\end{vmatrix}}{\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}}$$

Rank and Inverses

- Finding $\mathbf{A}^{-1}$ for an n x n matrix requires the solution of **Ax** = **b** n times, where **b** is one column of the unit matrix
- We cannot solve this equation unless rank **A** = n
- An n x n square matrix **A** will not have an inverse unless its rank equals its size
- An alternative statement is that **A** will not have an inverse if Det(**A**) = 0

Rank and Inverses II

- Recall the general result for the elements, b_{ij}, of **B** = $\mathbf{A}^{-1}$
- $b_{ij} = A_{ji}$/Det(**A**), where A_{ij} is the cofactor of a_{ij}
- We see that b_{ij} is not defined if Det **A** = 0
- $\mathbf{A}^{-1}$ does not exist if Det **A** = 0
- Det **A** = 0 for an n x n determinant shows that Rank **A** < n
- Det **A** ≠ 0 and rank **A** = n: two equivalent conditions for $\mathbf{A}_{(n \times n)}$ to have an inverse

Why Eigenvalues/Eigenvectors

- In electrical and mechanical networks, provides fundamental frequencies
- Shows coordinate transformations appropriate for physical problems
- Provides way to express network problem as diagonal matrix
- Transformations based on eigenvectors used in some solutions of **Ax** = **b**

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Eigenvalues and Eigenvectors

- Basic definition (**A** square)**: Ax** = λ**x**
- **x** is eigenvector, λ is eigenvalue
- Basic idea is that eigenvector is special vector of matrix **A**; multiplication of **x** by **A** produces **x** multiplied by a constant
- **Ax** = λ**x** => **Ax** – λ**x** = [**A** – **I**λ]**x** = **0**
- Homogenous equations; requires Det [**A** – **I**λ] = 0 for solution other than **x** = **0**

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Det[A – Iλ] = 0

$$Det[\mathbf{A}-\mathbf{I}\lambda] = Det\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn}-\lambda \end{bmatrix} = 0$$

- Det[**A** – **I**λ] = 0 produces an n[th] order equation that has n roots for λ. May have duplicate roots for eigenvalues.

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Two-by-two Matrix Eigenvalues

- Quadratic equation with two roots for eigenvalues

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda) - a_{21}a_{12} =$$

$$\lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{21}a_{12} = 0$$

- Eigenvalue solutions

$$\lambda = \frac{(a_{11}+a_{22}) \pm \sqrt{(a_{11}+a_{22})^2 - 4\underbrace{(a_{11}a_{22}-a_{21}a_{12})}_{\text{Det }\mathbf{A}}}}{2}$$

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Two-by-two Matrix Eigenvalues

- Write $\sqrt{(a_{11}+a_{22})^2 - 4Det\,\mathbf{A}}$ as $\sqrt{}$
- Add the two solutions to get

$$\lambda_1 + \lambda_2 = \frac{(a_{11}+a_{22}) + \sqrt{}}{2} + \frac{(a_{11}+a_{22}) - \sqrt{}}{2} = a_{11} + a_{22}$$

- Multiply the two solutions to get

$$\lambda_1\lambda_2 = \left(\frac{(a_{11}+a_{22}) + \sqrt{}}{2}\right)\left(\frac{(a_{11}+a_{22}) - \sqrt{}}{2}\right) = \frac{(a_{11}+a_{22})^2 - \left(\sqrt{}\right)^2}{4}$$

$$= \frac{(a_{11}+a_{22})^2 - (a_{11}+a_{22})^2 + 4Det\,\mathbf{A}}{4} = Det\,\mathbf{A}$$

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Sum and Product

- The results on the previous slide apply to all matrices
- The sum of the eigenvalues is the sum of the diagonal elements of the matrix, called the trace of the matrix
- The product of the eigenvalues is the determinant of the matrix

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = Trace\,\mathbf{A} \qquad \prod_{i=1}^{n} \lambda_i = Det\,\mathbf{A}$$

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Two-by-two Matrix Eigenvectors

- Two eigenvectors: $\mathbf{x}_{(1)} = [x_{(1)1}\ x_{(1)2}]^T$ and $\mathbf{x}_{(2)} = [x_{(2)1}\ x_{(2)2}]^T$ ($\mathbf{x}_{(j)} = [x_{(j)1}\ x_{(j)2}]^T$)
- Substitute each eigenvalue solution, λ_j, into $(\mathbf{A} - \mathbf{I}\lambda)\mathbf{x} = \mathbf{0}$ to find all $\mathbf{x}_{(j)}$ components

$$(a_{11} - \lambda_j)x_{(j)1} + a_{12}x_{(j)2} = 0$$
$$a_{21}x_{(j)1} + (a_{22} - \lambda_j)x_{(j)2} = 0$$

Notation: y_i is component i of vector $\mathbf{y}$; $\mathbf{z}_{(k)}$ is one of a vector set with components $z_{(k)i}$

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Two-by-two Eigenvectors II

- Eigenvector equations are homogeneous, so eigenvectors are determined only within a multiplicative constant
- Pick $x_{(j)1} = \alpha$ (arbitrary)

$$(a_{11} - \lambda_j)x_{(j)1} + a_{12}x_{(j)2} = 0$$
$$a_{21}x_{(j)1} + (a_{22} - \lambda_j)x_{(j)2} = 0$$

$$x_{(j)1} = \alpha \qquad x_{(j)2} = \frac{(\lambda_j - a_{11})}{a_{12}}\alpha = \frac{a_{21}}{(\lambda_j - a_{22})}\alpha$$

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How Many Eigenvalues?

- An n x n matrix has k ≤ n distinct eigenvalues
- Algebraic multiplicity of an eigenvalue, M_λ, is the number of roots of Det[**A** – **I**λ] = 0 that have the same root, λ
- Geometric multiplicity, m_λ, of eigenvalue is number of linearly independent eigenvectors for this λ

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Diagonalize a Matrix

- For an n x n matrix it is possible to create a matrix, **X**, where each column is one eigenvector
- One can then show that $\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \Lambda$, where Λ is a diagonal matrix whose components are the eigenvalues
  - Next assignment uses MATLAB to do this

$$X = \begin{bmatrix} x_{(1)1} & x_{(2)1} & \cdots & x_{(N)1} \\ x_{(1)2} & x_{(2)2} & \cdots & x_{(N)2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(1)N} & x_{(2)N} & \vdots & x_{(N)N} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \lambda_N \end{bmatrix}$$

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Supplemental Materials

- Items not planned for in-class coverage
  - Matrix equations for coordinate transforms
  - Derivation of 3x3 determinant formula from general determinant formula
  - Example of inverse calculation for 4x4 matrix (with many zeros)
  - Example of eigenvalue and eigenvector calculations
    - 2 x 2 array
    - 3 x 3 array

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Coordinate transformations

- Recall previous equations

$$y_1 = a_{11}x_1 + a_{12}x_2 \qquad z_1 = b_{11}y_1 + b_{12}y_2$$
$$y_2 = a_{21}x_1 + a_{22}x_2 \qquad z_2 = b_{21}y_1 + b_{22}y_2$$

- Define matrices so that **y** = **Ax** and **z** = **By**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

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Coordinate transformations II

- Show that matrix definitions give transformation results

$$y_1 = a_{11}x_1 + a_{12}x_2 \qquad z_1 = b_{11}y_1 + b_{12}y_2$$

$$y_2 = a_{21}x_1 + a_{22}x_2 \qquad z_2 = b_{21}y_1 + b_{22}y_2$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{By} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_{11}y_1 + b_{12}y_2 \\ b_{21}y_1 + b_{22}y_2 \end{bmatrix}$$

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Coordinate Transformations III

- From matrix equations **y** = **Ax** and **z** = **By**, we have **z** = **BAx** = **Cx** with **C** = **BA**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{Cx} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_{11}x_1 + c_{12}x_2 \\ c_{21}x_1 + c_{22}x_2 \end{bmatrix}$$

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Example of General Rule

- Get determinant of a 3 x 3 matrix by expansion along last row

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

$$A_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad A_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$A_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \qquad A_{31} = (-1)^{3+1} M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

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Example of General Rule II

- Get determinant of a 3 x 3 matrix

$$a_{31}M_{31} - a_{32}M_{32} + a_{33}M_{33} =$$

$$a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} =$$

$$= a_{31}(a_{12}a_{23} - a_{22}a_{13}) - a_{32}(a_{11}a_{23} - a_{21}a_{13}) + a_{33}(a_{11}a_{22} - a_{12}a_{21})$$

$$= a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{32}a_{11}a_{23} + a_{32}a_{21}a_{13} + a_{33}a_{11}a_{22} - a_{33}a_{12}a_{21}$$

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Example Problem

- Find **A**$^{-1}$ for **A** at right
- Have the following formula for **B** = **A**$^{-1}$

$$b_{ij} = (-1)^{i+j} \frac{M_{ji}}{Det(\mathbf{A})}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

  - General determinant formula: $\Sigma a_{ij}A_{ij}$
  - Take sum over third row to simplify calculation of Det **A**

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Example Problem Det **A**

- Det **A** = $(-1)^{3+1}a_{31}M_{31}$ + $(-1)^{3+2}a_{32}M_{32}$+ $(-1)^{3+3}a_{33}M_{33}$ + $(-1)^{3+4}a_{34}M_{34}$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

$$Det\ \mathbf{A} = (-1)^{3+1}(1)\begin{vmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{vmatrix} + 0 + 0 + 0$$

$$Det\ \mathbf{A} = \begin{vmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{vmatrix} = \begin{matrix} (0)(0)(1) + (1)(3)(1) + (2)(2)(0) \\ -(2)(0)(1) - (1)(2)(1) - (0)(3)(0) \end{matrix} = 1$$

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Example Problem III

- Apply: $b_{ij} = (-1)^{i+j}M_{ji}$/Det **A**
- Det **A** = 1 so $b_{ij} = (-1)^{i+j}M_{ji}$
  - Examples shown below

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

Remove for M_{21} Remove for M_{34}

$$b_{12} = (-1)^{1+2}M_{21} = -\begin{vmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$$b_{43} = (-1)^{4+3}M_{34} = -\begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 3 \end{vmatrix} = \begin{matrix} -(3+8+0 \\ -0-0-0) \end{matrix} = -11$$

Example Problem Solution

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ -1 & -2 & 5 & 1 \\ 3 & 4 & -11 & -2 \end{bmatrix}$$

- We can show that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- $(\mathbf{AA}^{-1})_{11} = 1\cdot0 + 0\cdot0 + 2\cdot(-1) + 1\cdot3 = 1$
- $(\mathbf{A}^{-1}\mathbf{A})_{43} = 3\cdot2 + 4\cdot0 + (-11)\cdot0 + (-2)\cdot3 = 0$
- Only 14 left to check

Two-by-two Example

- Find eigenvalues and eigenvectors of **A**

$$\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$$

$$Det[\mathbf{A} - \mathbf{I}\lambda] = \begin{vmatrix} 1-\lambda & 5 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$Det[\mathbf{A} - \mathbf{I}\lambda] = (1-\lambda)(2-\lambda) - (0)(5) = 0$$

- Solutions are $\lambda_1 = 2$ and $\lambda_2 = 1$

Two-by-two Example Continued

- Find $\mathbf{x}_{(1)}$ components for $\lambda_1 = 2$
- Solve $[\mathbf{A} - \lambda\mathbf{I}]\mathbf{x} = \mathbf{0}$ for $\mathbf{x}_{(1)}$ components

$$\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$$

$$(1-2)x_{(1)1} + 5x_{(1)2} = -x_{(1)1} + 5x_{(1)2} = 0$$

$$0x_{(1)1} + (2-2)x_{(1)2} = 0x_{(1)1} + 0x_{(1)2} = 0$$

- One equation in two unknowns
- Pick $x_{(1)2} = \alpha$ then $x_{(1)1} = 5\alpha$ from first equation
- Eigenvector $\mathbf{x}_{(1)}$ is $[5\alpha\ \alpha]^T$

Two-by-two Example Concluded

- Next find $\mathbf{x}_{(2)}$ components for $\lambda_2 = 1$
- Same as approach for finding $\mathbf{x}_{(1)1}$

$$\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$$

$$(1-1)x_{(2)1} + 5x_{(2)2} = 0x_{(2)1} + 5x_{(2)2} = 0$$

$$0x_{(2)1} + (2-1)x_{(2)2} = 0x_{(2)1} + 1x_{(2)2} = 0$$

- Both equations give $x_{(2)2} = 0$
- Pick $x_{(2)1} = \beta$ ($x_{(2)1}$ cannot be determined)
- $\mathbf{x}_{(2)} = [\beta\ \ 0]^T$

Check Two-by-two Example

$$\mathbf{Ax}_{(1)} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 5\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} (1)(5\alpha) + (5)(\alpha) \\ (0)(5\alpha) + (2)(\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} 10\alpha \\ 2\alpha \end{bmatrix} = 2\begin{bmatrix} 5\alpha \\ \alpha \end{bmatrix} = \lambda_1\mathbf{x}_{(1)}$$

$$\mathbf{Ax}_{(2)} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \beta \\ 0 \end{bmatrix} = \begin{bmatrix} (1)(\beta) + (5)(0) \\ (0)(\beta) + (2)(0) \end{bmatrix}$$

$$= \begin{bmatrix} \beta \\ 0 \end{bmatrix} = 1\begin{bmatrix} \beta \\ 0 \end{bmatrix} = \lambda_2\mathbf{x}_{(2)}$$

Eigenvector Factors

- 2 x 2 example showed $\mathbf{Ax}_{(j)} = \lambda_j\mathbf{x}_{(j)}$ regardless of choice of α and β
- This is general result
- We can pick one eigenvector component; typical choices are to make eigenvector simple or a unit vector

$$\mathbf{x}_{(1)} = \begin{bmatrix} 5\alpha \\ \alpha \end{bmatrix} \quad \mathbf{x}_{(1)} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad \mathbf{x}_{(1)} = \begin{bmatrix} 5/\sqrt{26} \\ 1/\sqrt{26} \end{bmatrix} \qquad \mathbf{x}_{(2)} = \begin{bmatrix} \beta \\ 0 \end{bmatrix} \quad \mathbf{x}_{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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Multiple Eigenvalue Example

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & -6 \\ 2 & -1 & -3 \\ -2 & -1 & 1 \end{bmatrix} \qquad \mathbf{A} - \mathbf{I}\lambda = \begin{bmatrix} 2-\lambda & 2 & -6 \\ 2 & -1-\lambda & -3 \\ -2 & -1 & 1-\lambda \end{bmatrix}$$

$$\begin{aligned} Det(\mathbf{A}-\mathbf{I}\lambda) &= (2-\lambda)(-1-\lambda)(1-\lambda) + (2)(-1)(-6) \\ &+ (-2)(2)(-3) - (-2)(-1-\lambda)(-6) - (2)(2)(1-\lambda) \\ &- (2-\lambda)(-1)(-3) = -\lambda^3 + 2\lambda^2 + \lambda - 2 + 12 + 12 + 12 \\ &+ 12\lambda - 4 + 4\lambda - 6 + 3\lambda = -\lambda^3 + 2\lambda^2 + 20\lambda + 24 = 0 \end{aligned}$$

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Multiple Eigenvalue Example II

$$Det(\mathbf{A}-\mathbf{I}\lambda) = (\lambda+2)(\lambda+2)(\lambda-6) = 0$$

- Solutions are λ = 6, –2, –2
- λ = –2 has algebraic multiplicity of 2
- Find eigenvector(s) from $(\mathbf{A} - \mathbf{I}\lambda_k)\mathbf{x}_{(k)} = \mathbf{0}$

$$\begin{bmatrix} 2-\lambda_k & 2 & -6 \\ 2 & -1-\lambda_k & -3 \\ -2 & -1 & 1-\lambda_k \end{bmatrix} \begin{bmatrix} x_{(k)1} \\ x_{(k)2} \\ x_{(k)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Look at λ_k = –2

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Multiple Eigenvalue Example III

$$\begin{bmatrix} 2-(-2) & 2 & -6 \\ 2 & -1-(-2) & -3 \\ -2 & -1 & 1-(-2) \end{bmatrix} \begin{bmatrix} x_{(k)1} \\ x_{(k)2} \\ x_{(k)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & -6 \\ 2 & 1 & -3 \\ -2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_{(k)1} \\ x_{(k)2} \\ x_{(k)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Apply Gauss elimination to these equations

$$\begin{bmatrix} 4 & 2 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{(k)1} \\ x_{(k)2} \\ x_{(k)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Pick $x_{(k)3}$ ***and*** $x_{(k)2}$

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Multiple Eigenvalue Example IV

$$\begin{bmatrix} 4 & 2 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{(k)1} \\ x_{(k)2} \\ x_{(k)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Pick $x_{(k)2}$ and $x_{(k)3}$ then $$x_{(k)1} = \frac{6x_{(k)3} - 2x_{(k)2}}{4}$$

- Pick $x_{(k)3}$ = 2 and $x_{(k)2}$ = 0 => $x_{(k)1}$ = 3
- Pick $x_{(k)3}$ = 0 and $x_{(k)2}$ = 2 => $x_{(k)1}$ = -1
- Two linearly independent eigenvectors for λ = -2

$$\mathbf{x}_{(1)} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{x}_{(2)} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

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Continue Example for λ_3 = 6

$$(\mathbf{A} - \mathbf{I}\lambda_3)\mathbf{x}_{(3)} = \begin{bmatrix} 2-6 & 2 & -6 \\ 2 & -1-6 & -3 \\ -2 & -1 & 1-6 \end{bmatrix} \begin{bmatrix} x_{(3)1} \\ x_{(3)2} \\ x_{(3)3} \end{bmatrix} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 & -6 \\ 2 & -7 & -3 \\ -2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_{(3)1} \\ x_{(3)2} \\ x_{(3)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Apply Gauss elimination to these equations

$$\begin{bmatrix} -4 & 2 & -6 \\ 0 & -6 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{(3)1} \\ x_{(3)2} \\ x_{(3)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Pick $x_{(3)3}$ =1 => $x_{(3)2}$ =-1

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Example Results

$$\begin{bmatrix} -4 & 2 & -6 \\ 0 & -6 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_{(3)1} \\ x_{(3)2} \\ x_{(3)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad x_{(3)1} = \frac{6x_{(3)1} - 2x_{(3)1}}{-4} = \frac{6(1) - 2(-1)}{-4} = -2$$

- Eigenvalues $\lambda_1 = -2$, $\lambda_2 = -2$, $\lambda_3 = 6$ have eigenvectors shown below

$$\mathbf{x}_{(1)} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{x}_{(2)} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{x}_{(3)} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

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Eigenvector Linear Dependence

- Can we have $\alpha_1\mathbf{x}_{(1)} + \alpha_2\mathbf{x}_{(2)} + \alpha_3\mathbf{x}_{(3)} = \mathbf{0}$ without $\alpha_1 = \alpha_2 = \alpha_3 = 0$?

$$\alpha_1\mathbf{x}_{(1)} + \alpha_1\mathbf{x}_{(2)} + \alpha_3\mathbf{x}_{(3)} = \alpha_1\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + \alpha_2\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \alpha_3\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Homogenous equations have $\alpha_1 = \alpha_2 = \alpha_3 = 0$ if matrix has full rank

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Linear Dependence II

- Matrix has full rank it its determinant is not zero

$$Det\begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{matrix} (3)(2)(1) + (0)(0)(-2) + \\ (2)(-1)(-1) - (2)(2)(-2) - \\ (0)(-1)(1) - (3)(0)(-1) = 15 \end{matrix}$$

- Since determinant is not zero, the only solution is $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so eigenvectors are linearly independent

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Quiz Three Solutions

1. Results of MATLAB commands

>> A = [1; 2; 3; 4] $\quad A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

>> B = [5 6; 7 8; 9 10] $\quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \\ 9 & 10 \end{bmatrix}$

>> C = [A B] Error: A and B must have the same number of rows

>> D = [A(2:4,1) B(:,2)] $\quad D = \begin{bmatrix} 2 & 6 \\ 3 & 8 \\ 4 & 10 \end{bmatrix}$

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Quiz Three Solutions II

2. Write MATLAB Commands

$E = \begin{bmatrix} 3 & 5 & -6 & 10 \\ 12 & 7 & -9 & 4 \end{bmatrix}$ E = [3 5 -6 10; 12 7 -9 4]

$F = \begin{bmatrix} 4 & 7 \\ -2 & 9 \\ 6 & 0 \end{bmatrix}$ F = [4 7; -2 9; 6 0]

3. Results of series of two commands

>> t = 2:6 >> x = log(t) .^ (t – 1)	$(\log(2))^1$, $(\log(3))^2$, $(\log(4))^3$, $(\log(5))^4$, $(\log(6))^5$
>> t = 0:pi/4:pi >> y = cos(t) ./ exp(t)	$\cos(0)/e^0$, $\cos(\pi/4)/e^{\pi/4}$, $\cos(\pi/2)/e^{\pi/2}$, $\cos(3\pi/4)/e^{3\pi/4}$,$\cos(\pi)/e^{\pi}$
>> t = 0:2:10 >> z = sum(t)	z = 0 + 2 + 4 + 6 + 8 + 10 = 30

Quiz Three Solutions III

4. Results of (a) G = [E; [F F]] and (b) H = [F; E']?

$$\boldsymbol{G} = \begin{bmatrix} 3 & 5 & -6 & 10 \\ 12 & 7 & -9 & 4 \\ 4 & 7 & 4 & 7 \\ -2 & 9 & -2 & 9 \\ 6 & 0 & 6 & 0 \end{bmatrix} \qquad \boldsymbol{H} = \begin{bmatrix} 4 & 7 \\ -2 & 9 \\ 6 & 0 \\ 3 & 12 \\ 5 & 7 \\ -6 & -9 \\ 10 & 4 \end{bmatrix}$$

5. G array after the command G(2:4,2:3) = F?

$$G = \begin{bmatrix} 3 & 5 & -6 & 10 \\ 12 & 4 & 7 & 4 \\ 4 & -2 & 9 & 7 \\ -2 & 6 & 0 & 9 \\ 6 & 0 & 6 & 0 \end{bmatrix}$$

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First Programming Assignment

- Number of students: 20
- Maximum possible score: 40
- Mean: 31.7
- Median: 34
- Standard deviation: 6.94
- Grade distribution:

18 21 23 24 25 26 26
30 32 32 36 36 37
37 37 37 38 39 39 40

Comments on Assignment

- Range of methods considered
- Cell formulas are simplest and quickest
- Use user defined functions (UDF) when you have repeated calculations and you want to avoid errors in reentering formulas
- Range names better identify variables
  - not so important for simple calculations, but useful for more complex workbooks

Comments on Assignment II

- Array function and macro require a lot of coding which is justified only if calculation is repeated by several users
  - With array function multiple tables can be placed on worksheet, each driven by separate input cells
    - Difficult to specify exact number of cells by selection
  - Macro requires recoding to use other inputs or place results in other locations
    - Uses exact specification of number of cells

Second Quiz

- Number of students: 22
- Maximum possible score: 25
- Mean: 15.9
- Median: 16.5
- Standard deviation: 6.63
- Grade distribution:

3 6 7 10 11 11 11
11 12 13 15 18 20 21 21
21 22 22 23 23 24 25

Comments on Second Quiz

- Confusion over transpose
  - MATLAB A' denotes transpose (usually A^T)
  - For $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$
- Expression cos(t).^(t-1) is same as (cos(t)).^(t-1) NOT cos[(t).^(t-1)
- Sum (x), where x is a 1D array gives the a sum of all elements in (For a 2D array it gives sum of each column

Second Program

- Number of students: 20
- Maximum possible score: 40
- Mean: 26.5
- Median: 27.5
- Standard deviation: 8.8
- Grade distribution:

10 10 12 19 19 24 24
24 27 27 28 30 31
32 34 34 34 35 38 38

Second Program Comments

- Do not generate or do not copy output or large arrays
- Differences between scripts and functions
  - Functions receive variable values through argument list, just as functions do in other languages like VBA
  - Variables in scripts, that are not set in the script, use the current value of the variable in the workspace

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MATLAB *vs*. Excel/VBA

- We have focused on use of MATLAB from command window
  - This is similar to entering data on the Excel spreadsheet
- Less focus on MATLAB programming
  - This is comparable to VBA
- Many students prefer Excel because of familiarity
- Excel/VBA more readily available, especially in small companies

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Quiz Three Solutions

- Initial guesses show that $x_+ = 3$ and $x_- = 1$ because f(3) > 0 and f(1) < 0

$$x_{new} = x_- - f(x_-)\frac{x_+ - x_-}{f(x_+) - f(x_-)}$$

$$= 1 - (-0.5403)\frac{3 - 1}{4.2858 - (-0.5403)} = 1.2239$$

- $f(x_{new}) < 0$ so x_{new} replaces x_-

$$x_{new} = 1.2239 - (-0.09269)\frac{3 - 1.2239}{4.2858 - (-0.09269)} = 1.2615$$

$$relErr = \left|\frac{1.2615 - 1.2239}{1.2615}\right| = 0.0298 > 0.001\text{: }\mathbf{continue}$$

- $f(x_{new}) < 0$ so x_{new} replaces x_-

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Quiz Three Solutions II

$$x_{new} = 1.2615 - (-0.01133)\frac{3 - 1.2615}{4.2858 - (-0.01133)} = 1.2661$$

$$relErr = \left|\frac{1.2615 - 1.2661}{1.2661}\right| = 0.00362 > 0.001\text{: }\mathbf{continue}$$

- $f(x_{new}) < 0$ so x_{new} replaces x_-

$$x_{new} = 1.2661 - (-0.01130)\frac{3 - 1.2661}{4.2858 - (-0.01130)} = 1.2666$$

$$relErr = \left|\frac{1.2666 - 1.2661}{1.2666}\right| = 0.000416 < .001\text{: }\mathbf{finished}$$

x = 1.2666

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Review Last Lecture

- Discussion of matrices: $a_{(row)(column)}$
  - Row, column, diagonal, square, unit(**I**), null(**0**)
  - **A** = **B**, **A** ± **B**, x**A**, **A**/x (scalar x), $\mathbf{A}^T$
  - Matrix multiplication: **P** = **LR**
    - Columns in **L** = rows in **R** = q $p_{ij} = \sum_{k=1}^{q} l_{ik} r_{kj}$
    - **P** rows = **L** rows; **P** columns = **R** columns
    - Multiply row of **L** by column of **R**
  - Determinants: single number for rectangular array; formula depends on array size
  - Inverse, $\mathbf{A}^{-1}$: $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

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Review Last Lecture II

- Use MATLAB commands det(A) and inv(A) for determinant and inverse
- Use Excel formulas mmult(A, B) for matrix multiplication, mdeterm(A) for determinant, and minverse(A) for inverse
  - minverse and mmult are array formulas
- Simultaneous linear equations in general matrix form, **Ax** = **b** $\sum_{j=1}^{N} a_{ij} x_j = b_i$
  - Variables x_1, x_2, x_3, …
  - Coefficients a_{ij} multiplies x_j in equation i
  - Right hand side b_i in equation i

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Review Last Lecture III

- Linear dependence: if one equation in a system of equations is a linear combination of one or more other equations, the system is said to be linearly dependent
  - A system that is not linearly dependent is **linearly independent**
  - In matrix form an equation is a row
- Matrix rank = number of linearly independent rows = number of linearly independent columns
- MATLAB formula rank(A) for matrix rank

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Review Last Lecture IV

- Solution of a system of linear algebraic equations, **Ax** = **b**, with n unknowns depends of rank of **A** and [**A b**]
  - Unique solution: rank(**A**) = rank([**A b**]) = n
  - Infinite solutions: rank(**A**) = rank([**A b**]) < n
  - No solutions: rank(**A**) ≠ rank([**A b**])
- Gaussian elimination: process to get matrix for system of equations in upper triangular form for simple back substitution
  - Also used to determine rank
  - Will review in detail later

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